Journal homepage www.jzs.uos.edu.krd
Journal of Zankoi Sulaimani
Part-A- (Pure and Applied Sciences)

On Simple Singular AGP-Injective Modules and AGP-Injective Rings with Some Types of Regular Rings

Abdullah M. Abdul-Jabbar

Department of Mathematics, College of Science, Salahaddin University-Erbil, Kurdistan Region Iraq
E-mail:abdullah.abduljabbar@su.edu.krd

Article info
Original: 27 Mar 2015
Revised: 1 May 2015
Accepted: 31 May 2015
Published online:
20 Dec. 2015

Key Words:
AP-injective modules,
AGP-injective modules,
AGP-injective rings,
regular rings.

Abstract

A ring R is called left AGP-injective if for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and a^nR is a direct summand of $r\ell(a^n)$. Now, in the present paper, we invistigate some properties of rings whose simple singular right R-modules are AGP-injective. Also, we give a characterization of π-regular rings interms of right weakly continuous ring whose simple singular right R-modules are AGP-injective under the condition, MERT ring. Finally, we give a property of AGP-injective rings with an index set $\{X_{a^n}: a \in R$ and n is a positive integer$\}$ of ideals such that $X_{a^n b} = X_{ba^n}$, for all a, b $\in$ R and a positive integer n.

1. Introduction and Preliminaries

Throughout this paper, R denotes an associative ring with identity and all modules are unitary. For a subset X of R, the right annihilator of X in a ring R is defined by $r(X) = \{t \in R: xt = 0, \text{ for all } x \in X\}$. Similarly, define the left annihilator of X in a ring R as $\ell(X) = \{ t \in R: tx = 0, \text{ for all } x \in X\}$. If $X = \{a\}$, we usually us to the abbreviation r(a) (resp. $\ell(a)$). An ideal I of a ring R is said to be essential if and only if I has a non-zero intersection with every non-zero ideal of R. Let R be a ring and x an element in R. Then x is said to be a right singular if and only if r(x) is an essential right ideal of R. The set of all right sigular elements in R is denoted by Y(R). Y(R) is a right ideal of R, which is called the right singular ideal of R. The left singular ideal Z(R) is similarly defined. A ring R is called right (left) non-singular if Y(R) = (0) (resp. Z(R) = (0)). For a left R-module M, $Z(M) = \{z \in M: \ell(z)$ is an essential left ideal of R$\}$ is called the left singular submodule of M. M is called left non-singular (resp. singular) if Z(M) = (0) (resp. Z(M) = M). The Jacobson radical [3] of a ring R, denoted by J(R), is the intersection of all maximal ideals of R. Recall that a ring R is called right principally injective [21] (or right P-injective for short) if every homomorphism from a principal right ideal of R to R can be extended to an endomorphism of R, or equivalently, $\ell r(a) = Ra$, for all $a \in R$. The concept of right P-injective rings has been generalized by many authors. For example, in [20, 26]. Following [20], a ring R is called right Generalized P-injective (briefly, GP-injective) if for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism from a^nR to R can be extended to an endomorphism of R. Note that GP-injective rings some times are also called YJ-injective in [18]. From [9], we know that GP-injective rings need not be P-injective. Recall that a ring R is said to be reduced if it has no non-zero nilpotent elements, or equivalently, $a^2 = 0$ implies $a = 0$, for all $a \in R$. Recall that a ring R is called left Kasch if $r(M) \neq (0)$, for every maximal left ideal M of R. An element a of a ring R is said to be regular [28] if there exists an element $b \in R$ such that $a = aba$. A ring R is said to be von Neumann regular (or, just regular) if every element of R is regular. A ring R is called π-regular [15] if for every element $a \in R$, there exists $b \in R$ and a positive integer n such that $a^n = a^nba^n$. A ring R is said to be strongly π-regular [5] if for every $a \in R$, there exists a positive integer n and $x \in R$ such that $a^n = a^{n+1}x$. Recall that a non-zero in a ring will be called right uniform if it contains no direct sum of right ideals. $X \leq M$ denote that X is a submodule of M. As defined by Utumi, a ring R is called left continuous if (i) every left ideal of R is essential in a direct summand of R and (ii) every left ideal isomorphic to a direct summand of R is itself a direct summand [27].

2. Simple Singular AGP-Injective Modules

In this section, we investigate some properties of rings whose simple singular right R-modules are AGP-injective. Also, we give a characterization of π-regular rings interms of right weakly continuous ring whose simple singular right R-modules are AGP-injective under the condition, MERT-ring. Finally, we give some equivalent statements of strongly π-regular rings interms of AGP-injective rings under the condition, ZI ring.

Following [24], the ring R is left AP-injective if for any $a \in R$, aR is a direct summand of $r\ell(a)$, and R is left AGP-injective if for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and a^nR is a direct summand of $r\ell(a^n)$.

Every left P-injective ring is left AP-injective and every left GP-injective ring is left AGP-injective. The rings R in [25, Examples 2.3, 2.4] are commutative AP-injective rings, but not GP-injective.

Recall a module M_R with $S = End(M_R)$ is said to be almost principally injective (or AP-injective for short) [24], if for any $a \in R$, there exists a left S-submodule X_a of M_R such that $\ell_M r_R(a) = Ma \oplus X_a$, where $\ell_M r_R(a)$ consists of all elements $z \in R$ such that $ax = 0$ implies $zx = 0$, for any $x \in R$.

We start this section with the following lemma.

Lemma 2.1: Let M be a right R-module with S = End(M_R). If $\ell_M r_R(a^n) = Ma^n \oplus X_{a^n}$ for every a ∈ R and a positive integer n, where X_{a^n} is a left S-submodule of M_R. Set f: $a^nR \rightarrow M$ is a right R-homomorphism, then $f(a^n) = m\, a^n + x$ with m ∈ M and $x \in X_{a^n}$.

Proof: Since $f(a^n)\, r_R(a^n) = f(a^n\, r_R(a^n)) = f(0) = 0$, for every a ∈ R and a positive integer n, so $r_R(a^n) \subseteq r_R(f(a^n))$, thus $\ell_M r_R(f(a^n)) \subseteq \ell_M r_R(a^n) = Ma^n \oplus X_{a^n}$ and $f(a^n) \in \ell_M r_R(f(a^n))$, hence $f(a^n) = ma^n + x$ with $m \in M_R$ and $x \in X_{a^n}$.

Recall that a ring R is called right (resp. left) weakly π-regular [12] if for every a ∈ R, there exists a positive integer n = n(a), depend on a such that $a^n \in a^nRa^nR$ (resp. $a^n \in Ra^nRa^n$). A ring R is called weakly π-regular if it is both right and left weakly π-regular.

Theorem 2.2: If every simple right R-module is AGP-injective, then R is a right weakly π-regular.

Proof: We will show that $R\, a^n\, R + r(a^n) = R$, for every a ∈ R and a positive integer n. Suppose that there exists b ∈ R such that $R\, b^n\, R + r(b^n) \neq R$, for every positive integer n, then there exists a maximal right ideal M of R containing $R\, b^n\, R + r(b^n)$. Thus, R / M is AGP-injective, then $\ell_{R/M} r_R(b^n) = (R/M)\, b^n \oplus X_{b^n}$, $X_{b^n} \leq R / M$. Let $f: b^nR \rightarrow R / M$ be defined by $f(b^nt) = t + M$, for every t ∈ R and a positive integer. Note f is a well-defined. So, $1 + M = f(b^n) = cb^n + M + x$, for c ∈ R and $x \in X_{b^n}$, then $1 - cb^n + M = x \in R / M \cap X_{b^n} = (0)$, $1 - cb^n \in M$, $cb^n \in M$ and hence 1 ∈ M, which is a contradiction. Therefore, $R\, a^n\, R + r(a^n) = R$, for every a ∈ R and a positive integer n, then R is a right weakly π-regular.

Recall that (1) R is called right (resp. left) duo [6] if every right (resp. left) ideal is a two-sided ideal; (2) R is called right quasi-duo [34] if every maximal right ideal of R is a two-sided ideal; (3) R is said to be weakly right duo [33] if for any a ∈ R, there exists a positive integer n such that a^nR is a two-sided ideal. Right quasi-duo (weakly right duo) rings are non-trivial generalizations of right duo rings (see [6, 32]). Note that if R right quasi-duo (weakly right duo), then R / J(R) is reduced (see [17]).

To prove Theorem 2.7 and Corollary 2.8, we need the following results.

Lemma 2.3 [4]: Every strongly π-regular ring is a right weakly π-regular.

A right R-module M is a regular module [36] if each of the following equivalent conditions hold:

(1) Every cyclic submodule of M is a direct summand.

(2) Every finitely generated submodule of M is a direct summand.

Recall that a right R-module M_R is said to have the exchange property [29] if for every module A_R and any two decompositions of AR,

$AR = M_R' \oplus NR = \oplus_{i\in I} Ai$ with $M_R' \cong MR$, there exist submodules $A_i' \subseteq Ai$ such that $AR = M_R' \oplus (\oplus_{i\in I} A_i')$.

MR is said to have the finite exchange property if the above condition is satisfied, whenever the index set I is finite.

Warfield [29] introduced the class of exchange rings. He called a ring R is an exchange rings if the right regular module R has the finite exchange property. After this Nicholson [22] proved that a ring R is exchange if and only if for each a ∈ R, there exists an idempotent e ∈ Ra such that (1-e) ∈ R (1-a).

Lemma 2.4 [7]: Any abelian exchange ring is quasi-duo.

Theorem 2.5 [1]: Let R be an abelian exchange ring. Then, R is a right weakly π-regular if and only if R is strongly π-regular.

Theorem 2.6: Let R be a quasi-duo. Then, R is a right weakly π-regular if and only if R is strongly π-regular.

Proof: It is obvious, directly in the same way of Theorem 2.5.

Theorem 2.7: If R is a right quasi-duo ring, then the following statements are equivalent:

(1) Every right R-module is AGP-injective;
(2) Every cyclic right R-module is AGP-injective;
(3) Every simple right R-module is AGP-injective;
(4) R is a strongly π-regular ring;
(5) R is a π-regular ring;
(6) R is a right weakly π-regular ring.

Proof: Obviously (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (1).

(4) $\Leftrightarrow$ (6). Follows from Theorem 2.6.

Thus, it remains to prove that (3) $\Rightarrow$ (4). For any $0 \neq a \in R$ and a positive integer n, we will show $a^n R + r(a^n) = R$. Suppose not. Then, there exists a maximal right ideal M of R containing $a^n R + r(a^n)$. Since R / M is simple, $\ell_{R/M} r_R (a^n) = (R/M) a^n \oplus X_{a^n}$, $X_{a^n} \leq R/M$. Let $f: a^n R \to R/M$ be defined by $f(a^n t) = t + M$, for every $t \in R$ and a positive integer n. Note that f is a well-defined. Thus, there exists $c \in R$ and $x \in X_{a^n}$ such that $1 + K = f(a^n) = ca^n + M + x$, then $1 - ca^n + M = x \in R/M \cap X_{a^n} = (0)$, thus $1 - ca^n \in M$. Since R is a right quasi-duo, then $ca^n \in M$, thus $1 \in M$, which is a contradiction. Therefore, $a^n R + r(a^n) = R$. So, R is strongly π-regular ring.

Corollary 2.8: If R is an abelian exchange ring, then the following statements are equivalent:

(1) Every right R-module is AGP-injective;
(2) Every cyclic right R-module is AGP-injective;
(3) Every simple right R-module is AGP-injective;
(4) R is a strongly π-regular ring;
(5) R is a π-regular ring;
(6) R is a right weakly π-regular ring.

Proof: Follows from Lemma 2.4.

Recall that R is a ZI ring [11] if for a, b ∈ R, ab = 0 implies aRb = 0.

Recall that a ring R is called MERT(resp. MELT) [16] if every maximal essential right (resp. left) ideal of R is a two-sided ideal.

Theorem 2.9: If R is a ZI ring, then the following statements are equivalent:

(1) R is a strongly π-regular ring;

(2) R is an MELT ring whose every simple left R-module is AGP-injective;

(3) R is an MERT ring whose every simple right R-module is AGP-injective;

(4) R is an MELT ring whose every simple singular left R-module is AGP-injective;

(5) R is an MERT ring whose every simple singular right R-module is AGP-injective.

Proof: Obviously (1) $\Rightarrow$ (3) $\Rightarrow$ (5) and (1) $\Rightarrow$ (2) $\Rightarrow$ (4).

(4) $\Rightarrow$ (1) and (5) $\Rightarrow$ (1) are similar.

We need only to prove (4) $\Rightarrow$ (1). Suppose (4), for any $0 \neq a \in R$ and a positive integer n, we will show that $Ra^n + \ell(a^n) = R$. If not, then there exists a maximal left ideal M of R containing $Ra^n + \ell(a^n)$, and M is an essential left ideal of R, thus R / M is left AGP-injective, then as the proof in Theorem 2.7, R is strongly π-regular.

Theorem 2.10: The following statements are equivalent:

(1) R is a strongly π-regular;

(2) R is a weakly right duo ring whose simple singular right R-modules are AGP-injective;

(3) R is an abelian right quasi-duo ring whose simple singular right R-modules are AGP-injective.

Proof: (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1). Since every weakly right duo ring is a right quasi-duo ring. Therefore, by Theorem 2.7, we get the result.

(1) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1). R is a right quasi-duo ring and R is abelian by [33, Lemma 4]. Thus, R is right weakly regular by Theorem 2.3 and hence R is right weakly π-regular ring. Hence R is a strongly π-regular ring by Theorem 2.6.

Theorem 2.11 [2]: The following statements are equivalent:

(1) R is π-regular;

(2) Every right R-module is GP-injective;

(3) Every cyclic right R-module is GP-injective.

Theorem 2.12: If R is an abelian right quasi-duo (inparticular, weakly right duo), then the following statements are equivalent:

(1) Every singular right R-module is AGP-injective;

(2) Every cyclic singular right R-module is AGP-injective;

(3) Every simple singular right R-module is AGP-injective;

(4) R is a strongly π-regular ring;

(5) R is a π-regular ring;

(6) R is a right weakly π-regular ring.

Proof: The trivial implications are (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) $\Rightarrow$ (6).

By Theorem 2.10, (3) $\Rightarrow$ (4).

(6) $\Leftrightarrow$ (4). Follows from Theorem 2.6.

If R is π-regular, then every right R-module is GP-injective by Theorem 2.11, so every right R-module is AGP-injective, thus (5) $\Rightarrow$ (1). This completes the proof.

Recall that a ring R is called right weakly continuous [23] if J(R) = Y(R), R / J(R) is regular and idempotents can be lifted modulo J(R) (i.e., if whenever $a^2 - a \in J(R)$, there exists $e^2 = e \in R$ such that $e - a \in J(R)$.

The following result is a characterization of π-regular rings interms of right weakly continuous ring whose simple singular right R-modules are AGP-injective under the condition, MERT ring.

Finally, we give he following main result.

Theorem 2.13: For an MERT ring R, the following statements are equivalent.

(1) R is π-regular;

(2) R is a right weakly continuous ring whose simple singular right R-modules are AGP-injective.

Proof: (1) $\Rightarrow$ (2). Observe that if R is π-regular, then by Theorem 2.7, every right R-module is AGP-injective. So, we are done.

(3) $\Rightarrow$ (1). Suppose that Y(R) $\neq$ (0). Then by [24, Lemma 1], we may assume that Y(R) is not reduced. So, there exists a non-zero $a \in Y(R)$ such that $a^2 = 0$. We claim that $Y(R) + r(a^n) = R$, for some positive integer n. If not, there exists a maximal essential right ideal M containing $Y(R) + r(a^n)$. Thus, R / M is AGP-injective, then $\ell_{R/M} r_R(a^n) = (R/M)a^n \oplus X_{a^n}$, $X_{a^n} \leq R/M$. Let f: $a^nR \rightarrow R/M$ be defined by $f(a^nt) = t + M$, for every $t \in R$ and a positive integer n. By Lemma 2.1, $1 + M = f(a^n) = ca^n + M + x$, for $c \in R$ and $x \in X_{a^n}$ then $1 - ca^n + M = x \in R/M \cap X_{a^n} = (0)$, so $1 - ca^n \in M$. Since R is an MERT ring, then $ca^n \in M$. Hence, $1 \in M$, which is a contradiction. Therefore, $Y(R) + r(a^n) = R$. Thus, we can write $1 = c + d$, for some $c \in Y(R)$ and $d \in r(a^n)$. Thus, $a^n = ca^n$ and so, $(1-c)a^n = 0$. Since $c \in Y(R) = J(R)$, 1-c is invertible. Thus, a = 0, which is a contradiction. Therefore, Y(R) is reduced and so Y(R) = (0).

3. A Connection Between AGP-Injective Rings with Some Types of Rings

In this section, we give some properties of AGP-injective rings. Moreover, we give some equivalent statements of left continuous regular rings via AGP-injective rings. Finally, we give a property of AGP-injective ring with an index set $\{X_{a^n}$: $a \in R$ and a positive integer n$\}$ of ideals such that $X_{a^nb} = X_{ba^n}$, for all $a, b \in R$ and a positive integer n.

We start this section with the following lemma.

Lemma 3.1: If R is left AGP-injective ring, then any left ideal isomorphic to a direct summand of $_RR$ is a direct summand of $_RR$.

Proof: Directly follows from [31, Lemma 2.1].

The following result is a characterization of left continuous regular ring.

Theorem 3.2: The following statements are equivalent for a ring R:

(1) R is a left continuous regular;

(2) R is a right AGP-injective ring such that for every cyclic left R-module M, $_R(M / Z(M))$ is projective;

(3) R is a left AGP-injective ring such that for every cyclic left R-module M, $_R(M / Z(M))$ is projective.

Proof: (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3).

By [19, Theorem 13] and all regular rings are both left and right AP-injective and hence it is AGP-injective.

(2) $\Rightarrow$ (1). By assumption, R / Z(R) is projective, which implies Z(R) is a direct summand of $_RR$, whence Z(R) = (0) (since Z(R) contains no no-zero idempotent elements). Then for every complement left ideal K of R, Z(R / K) = (0). By assumption a gain, left R-module R / K is projective, which implies $_RK$ is a direct summand of $_RR$. Since R is right AGP-injective, there exists a left ideal L of R and a positive integer n such that $\ell(r(a^n)) = Ra^n \oplus L$. Note that Z(R) = (0), which implies that $\ell(r(a^n))$ is a direct summand of $_RR$. This shows that Ra^n is a direct summand of $_RR$. Therefore, R is left continuous regular.

(3) $\Rightarrow$ (1). By Lemma 3.1 and apply the proof in (2) $\Rightarrow$ (1).

The following result is a property of reduced left AGP-injective rings.

Theorem 3.3: If R is a reduced left AGP-injective rings, then for any $a \in R$, $a \in a^{n+1}R$, for some positive integer n.

Proof: For any $0 \neq a \in R$ and a positive integer n, $\ell(a) = \ell(a^n)$ since R is reduced. Thus, there exists a right ideal L of R such that $a \in r(\ell(a)) = r(\ell(a^n)) = a^nR \oplus L$. Then, $a = a^n t + x$, for some $t \in R$ and $x \in L$, which implies $a^2 - a^{n+1} ta = xa \in a^nR \cap L = (0)$. Hence, $a^2 = a^{n+1} ta$, so $a = a^{n+1} t$ since R is reduced. This completes the proof.

Lemma 3.4: Let $c \in C(R)$, where C(R) is the center of R. If c is π-regular in R, then c is π-regular in C(R).

Proof: Let $c^n = c^n d c^n$ with $d \in R$ and a positive integer n. Set $u = d c^n d$. Then, $c^n = c^n u c^n = u c^{2n}$. We claim that $u \in C(R)$. In fact, for any $x \in R$, $ux - xu \in r(c^{2n}) = r(c^n)$, so $c^{2n} (xd^2 - d^2x) = c^n (xu - ux) = 0$, which implies $xd^2 - d^2x \in r(c^{2n}) = r(c^n) = 0$. Thus, $ux - xu = x c^nd^2 - c^nd^2x = c^n(xd^2 - d^2x) = 0$. This completes the proof.

Theorem 3.5: If R is a right nonsingular right AGP-injective ring, then the center C(R) of R is π-regular.

Proof: By hypothesis, R has a regular maximal right quotient ring S [10, Corollary 2.31] and hence R has π-regular maximal right quotient ring and hence the center C(S) of S is π-regular by Lemma 3.4.

For any $0 \neq a \in C(R)$, there exists $s \in S$ and a positive integer n such that $a^n = a^n s a^n = a^{2n}s = sa^{2n}$. Thus, $r(a^{2n}) = r(a^n) = r(a) = \ell(a) = \ell(a^n) = \ell(a^{2n})$, for any positive integer n by [30, Proposition 2.2].

We claim that a is π-regular in C(R). Note that $a^2 \neq 0$, so there is a positive integer m with $a^{2m} \neq 0$ such that $\ell r(a^{2m}) = Ra^{2m} \oplus X_{a^{2m}}$ for some left ideal $X_{a^{2m}}$ of R since R is right AGP-injective by [30, Proposition 2.2]. Thus, $a^{2m-1} \in \ell r(a^{2m-1}) = \ell r(a^{2m})$ and so $a^{2m-1} = da^{2m} + x$, for some $d \in R$ and some $x \in X_{a^{2m}}$. Then, $a^{2m} = a^m da^{2m} + a^m x$ and $a^m x \in Ra^{2m} \cap X_{a^{2m}} = (0)$. Hence, $a^{2m} = a^m da^{2m}$. Therefore, $1 - a^m d \in \ell(a^{2m}) = \ell(a^m)$, and so $a^m = a^m da^m$. This implies that C(R) is π-regular by Lemma 3.4.

Theorem 3.6 [30]: If R is a semiprime right AGP-injective ring, then the center C(R) of R is regular.

Corollary 3.7: If R is a semiprime right AGP-injective ring, then the center C(R) of R is π-regular.

Let e be an idempotent element of R. If eRe is a local ring, then e is called a local idempotent.

It is well known that local idempotents are primitive, but the converse is not true. For the integral ring **Z**, 1 is a primitive idempotent, but it is not local idempotent since **Z** is not a local ring, where **Z** is the set of all integer numbers.

Recall that a ring R is called orthogonally finite [30] if R has no infinite subsets consisting of orthogonal idempotents; R is called primitively finite if there exists finite orthogonal primitive idempotents $e_1, e_2, \ldots, e_n$ such that $1 = e_1 + e_2 + \ldots + e_n$ [30].

It is well known that R is semiperfect if and only if 1 is a sum of finite orthogonal local idempotents. Thus, every semiperfect ring is primitively finite, but the converse is not true (see Example 1 in [30] for the converse part).

Theorem 3.8 ([30, Theorem 2.9]): If R is a primitively finite right AGP-injective ring, then $R \cong R_1 \times R_2$, where R_1 is semisimple and every simple right ideal of R_2 is nilpotent.

Theorem 3.9: If R is semiperfect right AGP-injective ring, then $R \cong R_1 \times R_2$, where R_1 is semisimple and every simple right ideal of R_2 is nilpotent.

Proof: Follows from Theorem 3.8 and the fact that every semiperfect ring is primitively finite.

Following [8], a ring R is called generalized π-regular if for any $a \in R$, there exists a positive integer m such that $a^m = a^m ba$, for some $b \in R$.

For convenience, a ring R is said to be left generalized π-regular if for any $a \in R$, there exists a positive integer m such that $a^m = aba^m$, for some $b \in R$.

A ring R is said to satisfy the ascending chain condition [30] on the special right annihilators if for any $0 \neq x \in R$, the chain $r(x) \subseteq r(x^2) \subseteq \ldots \subseteq r(x^n) \subseteq \ldots$ terminates.

We can define the ascending chain condition for the left annihilators as follws:

Definition 3.10: A ring R is said to satisfy the ascending chain condition on the special left annihilators if for any $0 \neq x \in R$, the chain $\ell(x) \subseteq \ell(x^2) \subseteq \ldots \subseteq \ell(x^n) \subseteq \ldots$ terminates.

Following [37, Theorem 1.5], it is easy to verify that if R is a right AGP-injective ring satisfying the ascending chain condition on the special right annihilators, then J(R) is nilpotent.

Theorem 3.11: If R is a left AGP-injective ring satisfying the ascending chain condition on the special left annihilators, then R is right generalized π-regular.

Proof: Let $0 \neq a \in J(R)$, then there is a positive integer n such that $\ell(a^n) = \ell(a^{n+1})$ by hypothesis. If $a^n = 0$, then we are done. If $0 \neq a^n$ then $0 \neq a^{n+1}$ and so there is a positive integer m such that $0 \neq a^{m(n+1)}$ and $r(\ell(a^n)) = r(\ell(a^{m(n+1)})) = a^{m(n+1)}R \oplus X$ with $X \leq {}_R R$. Thus, $a^n = a^{m(n+1)} t + x$ with $t \in R$ and $x \in X$.

If m = 1, then $a^{n+1} = a^{n+1}ta$.

If m >1, then $a^{m(n+1)} = a^{m(n+1)} ta\, a^{(m-1)(n+1)}$.

In all cases, R is right generalized π-regular.

Corollary 3.12: If R is a left AGP-injective ring satisfying the ascending chain condition on the special left annihilators and $N_1 = \{0 \neq a \in R: a^2 = 0\}$ is regular (every element of N_1 is regular), then R is regular.

Proof: Follows from Theorem 3.11 and [8, Theorem 2.2].

Definition 3.13 [30]: A ring R satisfies left (resp. right) condition (*) if for any $0 \neq a \in R$, there exists a positive integer n such that $0 \neq a^n$ and Ra^n (resp. a^nR) is a pojective R-module.

Recall that a ring R is called right PP. ring if every principal right ideal of R is projective.

Clearly every right PP. ring satisfied right condition (*) [30].

Theorem 3.14: If R is a right AGP-injective ring and satisfies right condition (*), then R is generalized π-regular.

Proof: Suppose that R is a right AGP-injective ring satisfying right condition (*). Then, for any $0 \neq a \in R$, there exists a positive integer m with $0 \neq a^m$ such that $\ell r(a^m) = Ra^m \oplus X_{a^m}$, for some left ideal X_{a^m} of R. If we set $b = a^m$, then there is a positive integer p such that $0 \neq b^p$ and b^pR is projective. Thus, the short exact sequence $O \to r(b^p) \to R \to b^pR \to O$ splits. This implies that $r(b^p) = eR$ with $e^2 = e \in R$. Let $g = 1 - e$. Then $\ell r(b^p) = Rg$ and $g^2 = g \in R$, and so $b^p = b^pg$. Since $\ell r(b^p) \subseteq \ell r(b)$, $g = da^m + x$, for some $d \in R$ and some $x \in X_{a^m}$. Thus, $b^px = b^pg - b^pda^m = a^{mp} - a^{mp}da^m \in Ra^m \cap X_{a^m} = (0)$. This shows that $a^{mp} = a^{mp}da^m$, and so R is generalized π-regular.

Corollary 3.15: If R is right PP and right AGP-injective ring, then R is generalized π-regular.

Now, suppose an index set $\{X_{a^n}$: $a \in R$ and a positive integer n$\}$ of ideals such that $X_{a^n b} = X_{ba^n}$, for all $a, b \in R$ and a positive integer n.

Lemma 3.16: Given a set $\{X_{a^n}$: $a \in R$ and a positive integer n$\}$ of left ideals of R, the following are equivalent:

(1) $\ell r(a^n) = Ra^n \oplus X_{a^n}$, for all $a \in R$ and a positive integer n;

(2) $\ell[bR \cap r(a^n)] = (X_{a^n b} : b)_\ell + Ra^n$ and $(X_{a^n b} : b)_\ell \cap Ra^n \subseteq \ell(b)$, for all $a, b \in R$ and a positive integer n, where $(X_{a^n b} : b)_\ell = \{x \in R: xb \in X_{a^n b}\}$.

Proof: (1) $\Rightarrow$ (2). Let $x \in \ell[bR \cap r(a^n)]$, for some positive integer n. Then, $r(a^nb) \subseteq r(xb)$. So, $xb \in \ell r(xb) \subseteq \ell r(a^nb) = Ra^nb \oplus X_{a^n b}$. We write $xb = ta^nb + y$, where $t \in R$, $y \in X_{a^n b}$ and a positive integer n. Then, $(x - ta^n)b = y \in X_{a^n b}$ and hence $x - ta^n \in (X_{a^n b} : b)_\ell$. It follows that $x \in (X_{a^n b} : b)_\ell + Ra^n$.

It is obvious that $Ra^n \subseteq \ell[bR \cap r(a^n)]$. If $y \in (X_{a^n b} : b)_\ell$, then $yb \in X_{a^n b} \subseteq \ell r(a^nb)$. Let $bs \in bR \cap r(a^n)$. Then, $a^nbs = 0$. Hence $ybs = 0$ since $yb \in \ell r(a^nb)$. This shows that $y \in \ell[bR \cap r(a^n)]$. We have proved that $\ell[bR \cap r(a^n)] = (X_{a^n b} : b)_\ell + Ra^n$. If $ta^n \in (X_{a^n b} : b)_\ell + Ra^n$, then $ta^n b \in X_{a^n b} \cap Ra^n b$, showing that $ta^n b = 0$. Hence $ta^n \in \ell(b)$.

(2) ⇒ (1). Set b = 0, we obtain (1).

Theorem 3.17: Let R be a right AGP-injective ring with an index set $\{X_{a^n}$: a ∈ R and a positive integer n$\}$ of ideals such that $X_{a^n b} = X_{ba^n}$. If $(0) \neq uR$ is a uniform right ideal of R, define $M_u = \{x \in R: r(x) \cap uR \neq (0)\}$. Then M_u is the unique maximal left ideal of R which contains $\sum_{a\in R}(X_{a^n u}:u)_\ell$.

Proof: It is easy to see that M_u is a left ideal. Let $y \in (X_{a^n u}: u)_\ell$. Then, $ya \in X_{a^n u}$. Thus, $yua^n \in X_{a^n u} \cap Rua^n = X_{ua^n} \cap Rua^n$ since $X_{a^n u} = X_{ua^n}$ is an ideal, for some positive integer n. Then, $yua^n = 0$ and so $y \in M_u$ if $ua^n \neq 0$. If $ua^n = 0$, then $\ell r(ua^n) = 0$, and so $X_{a^n u} = X_{ua^n} = 0$. This shows that yu = 0 and hence $y \in M_u$. Therefore, $(X_{a^n u}: u)_\ell \subseteq M_u$ for all a ∈ R and a positive integer n. Now, if $a \notin M_u$, then $r(a^n) \cap uR = (0)$. By Lemma 3.16, we have $R = \ell(r(a^n) \cap uR) = (X_{a^n u}: u)_\ell + Ra^n$, for some positive integer n. Then, $R = M_u + Ra^n$, showing that M_u is a maximal left ideal.

Let L be a left ideal of R such that $\sum_{a\in R}(X_{a^n u}:u)_\ell \subseteq L \neq M_u$. Then, as above, $R = (X_{a^n u}: u)_\ell + Ra^n$, for any $a \in L - M_u$ and a positive integer n. Therefore, L = R.

The following result is a property of right AGP-injective ring with an index set $\{X_{a^n}$: a ∈ R and a positive integer n$\}$ of ideals such that $X_{a^n b} = X_{ba^n}$, for all a, b ∈ R and a positive integer n.

Finally, we give the following result.

Theorem 3.18: Let R be a right AGP-injective ring with an index set $\{X_{a^n}$: a ∈ R and a positive integer n$\}$ of ideals such that $X_{a^n b} = X_{ba^n}$, for all a, b ∈ R and a positive integer n. If R is left Kasch and R has a uniform right ideals, then M is a maximal left ideal of R if and only if $M = M_u$, for some uniform right ideal uR.

Proof: One direction is by Theorem 3.17. Let M be a maximal left ideal. Then $r(M) \neq (0)$ since R is left Kasch. Choose a uniform right ideal $uR \subseteq r(M)$. If $x \in M_u$, then $(0) \neq r(x^n) \cap uR \subseteq r(M)$, for some positive integer n. Thus, $R \neq \ell[r(x^n) \cap uR] \supseteq \ell r(M) \supseteq M$, implying $\ell[r(x^n) \cap uR] = M$. So, x ∈ M. By Theorem 3.17, $M_u = M$.

References

[1] Abdul-Jabbar A. M., "Certain Types of Weak Regularity", GP-injectivity and Flatness, Ph. D. Thesis, College of Science, Salahaddin University-Erbil. (2007).

[2] Al-Kouri M. A., "On π-Regular Rings", M. Sc. Thesis, Mosul University. (1996).

[3] Burton D. M., "A First Course in Rings and Ideals", Addison Wesley Publishing Company. (1970).

[4] Al-Kouri M. R. M., "On Strongly π-Regular Rings", Ph. D. Thesis, Mosul University. (2000).

[5] Azumaya G., "Strongly π-regular rings", J. Fac. Sci. Hokkaido Univ., Vol.13, pp. 34-39. (1954).

[6] Brown S. H., "Rings over which every simple modules is rationally complete", Canad J. Math.25, pp. 693-701. (1973).

[7] Chan Y. H., Nam K. K. and Lee Y., "Exchange rings and their extensions", J. of Pure and Applied Algebra, 179, pp. 117-126. (2003).

[8] Chen J. L. and Ding N. Q., "On regularity of rings", Algebra Colloquium, 4, pp. 267-274. (2001).

[9] Chen J., Zhou Y. and Zhu Z., "GP-injective rings need not be P-injective", Comm. Algebra, 33(7), pp. 2395-2402. (2005).

[10] Gooderl K. R., "Ring Theory: Nonsingular rings and modules", Pure Apl. Math. 33 Marcel Dekker, New York. (1991).

[11] Guangshi X., "On GP-V-rings and characterizations of strongly regular rings", Northeast. Math. J., 18(4), pp. 291-297. (2002).

[12] Gupta V., "Weakly π-regular rings and group rings", Mathematical Journal Okayama University, 19, pp. 123-127. (1977).

[13] Kim N. K., Nam S. B. and Kim J. Y., "On simple singular GP-injective modules", Comm. In Algebra, 27(5), pp. 2087-2096. (1999).

[14] Kim J. Y., "Certain rings whose simple singular modules are GP-injective", Proc. Japan Acad, 81, Ser. A, pp. 125-128. (2005).

[15] McCoy N. H., "Generalized regular rings", Bull. Amer. Math. Soc. 45, pp. 175-178. (1939).

[16] Ming R. Y. C., "On V-rings and prime rings", Journal of Algebra, 62, pp. 13-20. (1980).

[17] Ming R.Y.C., "On regular rings and Artinian rings", Riv. Mat. Univ. Parma., 4(11), pp. 101-109. (1985).

[18] Ming R. Y. C., "On injectivity and P-injectivity", Journal of Mathematics of Kyoto University, 27(3), pp. 439-452. (1987).

[19] Ming R. Y. C., "On injectivity and P-injectivity IV", Bull. Korean Math. Soc., 40(2), pp. 223-234. (2003)

[20] Nam S. B., Kim N. K., and Kim J. Y., "On simple GP-injective modules", Comm. Algebra, 23(14), pp. 5437-5444. (1995).

[21] Nicholson W. K. Yousif and M. F., "Principally injective rings", J. of Algebra, 174(1), pp. 77-93. (1995).

[22] Nicholson W. K., "Lifting idempotents and exchange rings", Trans. Amer. Math. Soc., 229, pp. 269-278. (1997).

[23] Nicholson W. K. and Yousif M. F., "Weakly continuous and C_2-rings", Comm. Algebra, 29(6), pp. 2429-2446. (2011).

[24] Page S. S. and Zhou Y. Q., "Generalizations of principally injective rings", J. of Algebra, 206, pp. 706-721. (1998).

[25] Page S. S. and Zhou Y., "Quasi-dual rings", Comm. Algebra, 28, pp. 489-504. (2000).

[26] Sanh N. V., Shum K. P., Dhompongsa S. and Wongwai S., "On quasi- principally injective modules", Algebra Colloquium, 6(3), pp. 269-276. (1999).

[27] Utumi Y., "On continuous rings and self-injective rings", Trans. Amer. Math. Soc., 118, pp. 158-173. (1965).

[28] von Neumann J., "On regular rings", Proc. Nat. Sci. U. S. A. 22, pp. 707-713. (1936)

[29] Warfield B., "Exchange rings and decompositions of modules", Math. Ann., 199, pp. 31-36. (1972).

[30] Xiao G., Ding N. and Tong W., "Regularity of AP-injective rings, Vietnam Journal of Mathematics", 32(4), pp. 399 - 411. (2004).

[31] Xiao G. and Tong W., "On injectivity and P-injectivity", Bull. Korean Math. Soc., 43(2), pp. 299-307. (2006)

[32] Xiao G. S., Yin X. B. and Tong W. T., "A note on AP-injective rings", J. Math. Research and Exposition 23, pp. 211-216. (2003)

[33] Yao X., "Weakly right duo rings", Pure and Applied Math. Sciences, 21, pp. 19-24. (1985).

[34] Yu H. P., "On quasi-duo ring", Glasgow Math. J., Vol. 37, pp. 21-31. (1995).

[35] Yu-e Z., "On simple singular AP-injective modules", International Mathematical Forum, 6(21), pp. 1037-1043. (2011).

[36] Zelmanowitz J., "Regular modules", Trans. Amer. Math. Soc., 163, pp. 341-355. (1972).

[37] Zhang J. L. and Wu J., "Generalizations of principal injectivity", Algebra Colloquium, 6(3), pp. 277-282. (1999).